<u>LAST TIME:</u> Given $\mathcal{C}^{\otimes}$ symm. mon. ∞-cat. a <u>commutative algebra</u> in $\mathcal{C}^{\otimes}$ is

$$A: *^{\mathrm{Fin}_*} \to \mathcal{C}^{\otimes, \mathrm{Fin}_*} \quad \text{which sends.}$$

inert coCartesian morphisms to inert coCartesian morphisms.

Explicitly, let $e_i: \langle 2\rangle \to \langle 1\rangle$ be $e_i(j) = \delta_{i,j}$ two inert morphisms.
then $A(e_i): A\langle 2\rangle \to A\langle 1\rangle$ is coCartesian if $\forall Z \in \mathcal{C}^{\otimes, \mathrm{Fin}_*}$ one has:

$$(\Delta) \quad \mathrm{Hom}_{\mathcal{C}^{\otimes}, \mathrm{Fin}_*}(A\langle 1\rangle, Z) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}^{\otimes}, \mathrm{Fin}_*}(A\langle 2\rangle, Z) \underset{\mathrm{Hom}_{\mathrm{Fin}_*}(\langle 2\rangle, p(Z))}{\times} \mathrm{Hom}_{\mathrm{Fin}_*}(\langle 1\rangle, Z).$$

Recall that ~~a~~ a morphism $A\langle 1\rangle \to Z$ in $\mathcal{C}^{\otimes, \mathrm{Fin}_*}$ is the data of:

$$f: \langle 1\rangle \to p(Z) \quad + \quad \tilde{f}: \mathcal{C}^{\otimes}(f)(A\langle 1\rangle) \to \bar{Z},$$

where $\bar{Z} \in \mathcal{C}^{\otimes}(p(Z))$.

Similarly, let $g: \langle 2\rangle \to p(Z) \ + \ \tilde{g}: \mathcal{C}^{\otimes}(g)(A\langle 2\rangle) \to \bar{Z}$ be an elmt on the right-hand side.
The condition that (Δ) is an isom. gives:

$$g = f \circ e_i \quad \text{and} \quad \tilde{g} \simeq \tilde{f} \circ \mathcal{C}^{\otimes}(e_i).$$

Since we require (Δ) holds for all $Z \in \mathcal{C}^{\otimes, \mathrm{Fin}_*}$, by considering $\bar{Z} \in \mathcal{C}^{\otimes}(\langle 1\rangle)$ and $f = \mathrm{id}_{\langle 1\rangle}$ one gets.

$$\mathrm{Hom}_{\mathcal{C}}(A\langle 1\rangle, \bar{Z}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}^{\otimes}(e_i)(A\langle 2\rangle), \bar{Z}) \quad \forall \bar{Z} \in \mathcal{C}.$$

$$\Rightarrow \mathcal{C}^{\otimes}(e_i)(A\langle 2\rangle) = A\langle 1\rangle.$$

Thus, by applying the description of $\otimes$ to the image of $A(\langle\alpha\rangle)$ we get:

$$A(\langle\alpha\rangle): A\langle 1\rangle \otimes A\langle 1\rangle \to A\langle 1\rangle.$$

However, notice that ~~we~~ requiring that $A(\langle\alpha\rangle)$ is $\otimes$ Cart. would imply $\mathcal{C}^{\otimes}(\alpha)(A\langle 2\rangle) \simeq A\langle 1\rangle$ in $\mathcal{C}$, which

$\underset{A\langle 1\rangle \otimes A\langle 1\rangle}{\overset{\text{is}}{}}$

we don't want to in general.

(Discussion of monoidal categories see last time. Maybe postponed.)

Here is another important example of a sym. mon. ∞-category, which is a sanity check for our definition.

Example: (ord. tensor cat.) Let C be a sym. mon. 1-cat. Consider the coCartesian fibration $C^{\otimes, \mathrm{Fin}_*} \to \mathrm{Fin}_*$ defined as follows:

- $C^{\otimes, \mathrm{Fin}_*}_{\langle n\rangle} = \{ X_i \in C \}_{1 \le i \le n}$.

a morphism

- $(\{X_i\}_{\langle n\rangle^\circ}, \langle n\rangle) \to (\{Y_j\}_{\langle m\rangle^\circ}, \langle m\rangle)$, is:

$$\alpha: \langle n\rangle \to \langle m\rangle \quad + \quad \tilde{\alpha} \in \prod_{j \in \langle m\rangle^\circ} \mathrm{Hom}_C\Big(\bigotimes_{i \in \alpha^{-1}(j)} X_i, Y_j\Big)$$

where $X_0 := 1_C$.

<u>Exercise:</u> Check the above gives a sym. mon. ∞-category.

<u>Sanity check 2:</u> For $C^{\otimes,\mathrm{Fin}_*} \to \mathrm{Fin}_*$ a sym. mon. ∞-cat. obtained as in the previous example.

$A \in \mathrm{CAlg}(C^{\otimes})$ a. comm. alg. object in the sense of last time.

is equivalent to $\bar{A} \in C$, + $\mu: \bar{A} \otimes \bar{A} \to \bar{A}$

a commutative an associative multiplication.

Indeed, the Remark from earlier today gives $A\langle 2\rangle \simeq A\langle 1\rangle \otimes A\langle 1\rangle$.

So $\bar{A} = A\langle 1\rangle$.

Associativity is the commutativity of the image of

$$\begin{pmatrix} 1 & 2,3 \\ \downarrow & \downarrow \\ 1 & 2 \end{pmatrix} \quad \begin{tikzcd} \langle 3\rangle \arrow[r, "{(1,2\mapsto 1,\ 3\mapsto 2)}"] \arrow[d] & \langle 2\rangle \arrow[d, "\alpha"] \\ \langle 2\rangle \arrow[r, "\alpha"'] & \langle 1\rangle \end{tikzcd}$$

. Similarly for commutativity.

That one can write $A: \cancel{\mathrm{Fin}_* \to \mathrm{Cat} \subset \mathrm{Cat}_\infty}$ $\mathrm{Fin}_* \to C^{\otimes,\mathrm{Fin}_*}$ fr. A is an easy check from the description. of $C^{\otimes,\mathrm{Fin}_*}$.

To compare the notions of derived rings to usual rings we need to quickly discuss one more notion about ∞-categories.

<u>Def'n:</u> Let $X \in \mathcal{C}$ be an object in an ∞-cat. X is said to be <u>n-truncated</u> $(n \geq 0)$ if $\forall A \in \mathcal{C}$ one has:

$$\mathrm{Hom}_{\mathcal{C}}(X, A) \in \mathrm{Spc}^{\leq n} = \{ S \in \mathrm{Spc} \mid \pi_i(S) = 0 \ \forall i > n \}.$$

Let $\tau_{\leq n}\mathcal{C}$ denote the subcategory of n-truncated objects.

In particular, objects of $\tau_{\leq 0}\mathcal{C}$ are called <u>discrete objects.</u>

Here are a couple of useful properties:

Lemma: (i) $\forall n \geq 0$, $\tau_{\leq n} \mathcal{C} \hookrightarrow \mathcal{C}$ is stable under limits. (that exist in $\mathcal{C}$)
(ii) $\tau_{\leq n} \mathcal{C} \to \mathcal{C}$ has a left adjoint.
(iii) any functor that preserves finite limits will preserve n-truncated objects.

Example: (i) $\tau_{\leq 0} Spc \simeq Sets.$
(ii) $\tau_{\leq 1} Spc = Grpd.$

To give more examples, we recall the following notion:

Def'n: (Cohomologically graded t-structure) Let D be a triangulated category. a t-structure on D is the data of two subcategories $(D^{\geq 0}, D^{\leq 0})$ s.t.:

(i) $D^{\leq -1} := [1](D^{\leq 0}) \subseteq D^{\leq 0}$ and $D^{\geq 1} := [-1](D^{\geq 0}) \subseteq D^{\geq 0}$;

(ii) $\forall$ $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$

$$\mathrm{Hom}_D(X, Y) \simeq 0.$$

(iii) $\forall X \in D$, $\exists$ $X_{con.} \in X^{\leq 0}$, and $X_{cocon.} \in X^{\geq 1}$ and a distinguished triangle:

$$X_{con.} \to X \to X_{cocon.}$$

For $\mathcal{C}$ a stable ∞-cat. a t-str. is the data of $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ subcategories s.t. $h\mathcal{C}$ $(h\mathcal{C}^{\leq 0}, h\mathcal{C}^{\geq 0})$ is a t-structure.

We give a small and incomplete list of properties:

<u>Prop:</u> (i) $\mathcal{C}^{\heartsuit} := \mathcal{C}^{\geq 0} \cap \mathcal{C}^{\leq 0}$ is a 1-category and it is abelian;

(ii) $\mathcal{C}^{\leq n} \hookrightarrow \mathcal{C}$ has a right adjoint $\tau^{\leq n}$, $\mathcal{C}^{\leq n}$ is stable under colimits;

(iii) $\mathcal{C}^{\geq n} \hookrightarrow \mathcal{C}$ has a left adjoint $\tau^{\geq n}$, $\mathcal{C}^{\geq n}$ is stable under limits.

The notations $\tau_{\leq n}$ and $\tau^{\geq -n}$ agree $\underline{\text{only}}$ on the subcategory $\mathcal{C}^{\leq 0}$.

<u>Claim:</u> $X \in \mathcal{C}^{\leq 0}$ is n-truncated. $= \mathrm{Hom}_{\mathcal{C}}(Y, X) \in \mathrm{Spc}_{\leq k}$ $\forall Y \in \mathcal{C}^{\leq 0}$.

$\Leftrightarrow \quad X \in \mathcal{C}^{\geq -k}$.

Notice $\pi_i(\mathrm{Hom}_{\mathcal{C}}(Y, X)) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(Y, X[i])) = \mathrm{Hom}_{h\mathcal{C}}(Y[-i], X)$.

$$\mathrm{Ext}^{-i}_{\mathcal{C}}(Y, X) = 0 \quad \forall Y \in \mathcal{C}^{\leq 0} \qquad \mathrm{Hom}_{h\mathcal{C}}(Y[-i], X) \overset{!!}{=} \mathrm{Ext}^{-i}(Y, X).$$

$$\|$$

$$\mathrm{Hom}_{h\mathcal{C}}(Y, X[-i]) = 0 \quad \forall Y \in \mathcal{C}^{\leq 0}.$$

$$\Downarrow$$

$X[-i] \in \mathcal{C}^{\geq 1} \Rightarrow X \in \mathcal{C}^{\geq 1-i}$, taking $\underline{i = k+1} \Rightarrow X \in \mathcal{C}^{\geq -k}$.

Property of a t-str. on a triangulated category.
$X \in h\mathcal{C}^{\geq 1}$
$\Updownarrow$
$\mathrm{Hom}_{h\mathcal{C}}(Y, X) = 0$ $\forall Y \in h\mathcal{C}^{\leq 0}$.

Thus, <u>Example:</u> (iii) For $\mathcal{C}$ stable w/ a t-structure:
$$\tau_{\leq n}(\mathcal{C}^{\leq 0}) \simeq \mathcal{C}^{\geq -n, \leq 0} := \mathcal{C}^{\geq -n} \cap \mathcal{C}^{\leq 0}.$$

As much as I am trying to avoid these notes have some circularity. For ease of exposition now we will assume a result, which we might "justify" later. ~~The loo~~

<u>FACT</u>: The ∞-category $\mathrm{Vect}_k := \mathcal{D}(k)$ has a sym. mon. structure (k will be a field for us, but this is true for any comm. ring).

It is characterized by: (i) $\otimes$ commutes w/ colimits in each variable.

(ii) when restricted to $\mathrm{Vect}_k^{\heartsuit} \simeq$ ordinary category of k-vector spaces: it agrees w/ the usual $\otimes$-product.

(Vect_k has a t-structure: $\mathrm{Vect}_k^{\leq 0} := \{ V \in \mathrm{Vect}_k \mid H^i(V) = 0 \ \forall i \geq 1 \}$

$\mathrm{Vect}_k^{\geq 0} := \{ \text{———} \mid H^i(V) = 0 \ \forall i \leq -1 \}$).

Notice that by <u>Ex:</u> (iii) above one has:

$$\tau_{\leq 0}(\mathrm{Vect}_k^{\leq 0}) \simeq \mathrm{Vect}_k^{\geq 0, \leq 0} = \mathrm{Vect}^{\heartsuit}$$ which is the ordinary cat. of k-vector spaces.

Moreover, b/c $\mathrm{Vect}^{\leq 0} \hookrightarrow \mathrm{Vect}$ preserves colimits, the $\otimes$-str. of Vect restricts to a $\otimes$-str. on $\mathrm{Vect}^{\leq 0}$.

Finally, we can define the first candidate for derived rings. (or k-algebras).

<u>Def'n:</u> A <u>connective differential graded k-algebra</u> (cdga) is an object of:

$$\mathrm{CAlg}_k := \mathrm{CAlg}(\mathrm{Vect}_k^{\leq 0}), \quad \cancel{\mathrm{CAlg}_k} := \mathrm{Fun}^{\text{right-lax}}_{\bullet}(\downarrow^{\mathrm{Fin}}, \mathrm{Vect}_k^{\leq 0, \otimes, \mathrm{Fin}}).$$

(we never formally wrote down what the category $\mathrm{CAlg}(\mathcal{C})$ for some sym. mon. ∞-cat. $\mathcal{C}$ is. The right-hand side above hints at what formal symbols can do this for us.)

Rk: Here is some justification for this definition.

First, we claim that $\tau^{\geq -n} : \mathrm{Vect}^{\leq 0} \to \mathrm{Vect}^{\geq -n, \leq 0}$ is a sym. monoidal functor.

Heuristically, notice that for any $V \xrightarrow{f} W$ in $\mathrm{Vect}^{\leq 0}$, s.t. $\tau^{\geq -n}(f)$ is an isomorphism one has:

$$\tau^{\geq -n}(V \otimes U) \xrightarrow{\sim} \tau^{\geq -n}(W \otimes U) \quad \text{for}$$

any $U \in \mathrm{Vect}^{\leq 0}$.

By applying this twice to $V \to \tau^{\geq -n}(V)$ and $U \to \tau^{\geq -n}(U)$ one gets:

$$\tau^{\geq -n}(V \otimes U) \xrightarrow{\sim} \tau^{\geq -n}(V) \otimes \tau^{\geq -n}(U).$$

Lemma: $\mathrm{Vect}^{\geq -n, \leq 0} \hookrightarrow \mathrm{Vect}^{\leq 0}$ is a right-lax functor.

This follows from the general fact justifying the names right/left lax functors.

Thus, given a comm. alg. object in $\mathrm{Vect}^{\geq 0, \leq 0}$, i.e. a classical commutative k-algebra one obtains an object in CAlg_k via:

$$\iota : \mathrm{CAlg}(\mathrm{Vect}^{\geq 0, \leq 0}) \longrightarrow \mathrm{CAlg}(\mathrm{Vect}^{\leq 0}). = \mathrm{CAlg}_k$$

Moreover, we claim:

Prop: ι is fully faithful with essential image the discrete objects of CAlg_k.

Idea: We haven't introduced these objects but given a free alg. functor: $\mathrm{Sym}: \mathrm{Vect}_k^{\leq 0} \to \mathrm{CAlg}_k$, this follows fr. the adjunction $\mathrm{Hom}_{\mathrm{CAlg}_k}(\mathrm{Sym}(V), A) \simeq \mathrm{Hom}_{\mathrm{Vect}^{\leq 0}}(V, \mathrm{oblv}(A))$

$\mathrm{oblv}: \mathrm{CAlg}_k \to \mathrm{Vect}^{\leq 0}$ forgets A is an alg.